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Noncommutative Cyclic Characters of Symmetric Groups

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We define noncommutative analogues of the characters of the symmetric group which are induced by transitive cyclic subgroups (cyclic characters). We investigate their properties by means of the formalism of noncommutative symmetric functions. The main result is a multiplication formula whose commutative projection gives a combinatorial formula for the resolution of the Kronecker product of two cyclic representations of the symmetric group. This formula can be interpreted as a multiplicative property of the major index of permutations.

1. Introduction

There is a well-known relation between symmetric functions and characters of symmetric groups. Recently, starting from the quasi-determinants of Gelfand and Retakh [5, 6] a noncommutative theory of symmetric functions has been developed [4] in such a way that most of the classical application can be lifted to the noncommutative case. In particular, the character theory of the symmetric group has a natural noncommutative analogue, in wich the rôle of the character ring is played by Solomon's descent algebra [16]. In this setting, the simplest noncommutative analogues of the classical calculations with characters can be interpreted in terms of certain idempotents

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of the descent algebra, which arise in various problems related to free Lie algebras (e.g., the study of the Hausdorff series; see [13, 14, 2]). The first applications of the descent algebra to character computations are due to Gessel [7].

In this paper, we study noncommutative analogues of certain symmetric functions, which are known to display a rich combinatorial structure. These are the Frobenius characteristics of the characters induced by transitive cyclic subgroups. These characters seem to have been first studied by Foulkes [3]. A combinatorial formula for their decomposition into irreducibles appears in [9] and is generalized in [10] to certain plethysms involving the same characters. Another generalization appears in [17]. Applications to free Lie algebras are collected, for example, in [14], and some other properties are discussed in [15].

This paper is organized as follows. Sections 2 and 3 provide the necessary background on noncommutative symmetric functions and cyclic characters. Then, the noncommutative cyclic characters $L_n^{(k)}$ are defined, together with another basis $(K_n^{(j)})$ of their linear span, and multiplicative properties of the $K_n^{(j)}$ are established (Section 4). It is then found that the $K_n^{(j)}$ and, thus, also the $L_n^{(k)}$ span a subalgebra, and the structure constants in these two bases are computed (Section 5). This gives in particular an explicit formula for the Kronecker product of two cyclic representations of the symmetric group. Finally, this result is given a combinatorial interpretation, in terms of a multiplicative property of the major index of permutations (Section 6), and a generalization to a class of nontransitive cyclic subgroups is sketched (Section 7).

Our notations for commutative symmetric functions are those of [11]. For the noncommutative ones, we use those of [4], which are recalled in the next section.

2. Noncommutative Symmetric Functions

The *algebra of formal noncommutative symmetric functions* is the free associative algebra $\mathbf{Sym} = \mathbf{C}\langle \Lambda_1, \Lambda_2, \dots \rangle$ generated by an infinite sequence of noncommuting indeterminates Λ_k, called the *elementary functions*. It is convenient to set $\Lambda_0 := 1$. Let t be another indeterminate, commuting with the Λ_k. One introduces the generating series

$$\lambda(t) = \sum_{k \geq 0} t^k \Lambda_k, \qquad \sigma(t) = \sum_{k \geq 0} t^k S_k = \lambda(-t)^{-1},$$

$$\Phi(t) = \sum_{k \geq 1} \frac{1}{k} \Phi_k t^k = \log \sigma(t).$$

The S_k are called *complete symmetric functions*, and the Φ_k are the *power sums of the second kind*. The *power sums of the first kind*, denoted by Ψ_k are the coefficients of the formal series

$$\psi(t) = \sum_{k \geqslant 1} t^{k-1} \Psi_k,$$

defined by the equation

$$\psi(t) = \lambda(-t) \frac{d}{dt} \sigma(t).$$

The algebra **Sym** is graded by the weight function w defined by $w(\Lambda_k) = k$, and its homogeneous component of weight n is denoted by $\mathbf{Sym}_n$. If (F_n) is a sequence of noncommutative symmetric functions such that $F_n \in \mathbf{Sym}_n$, we set for any composition $I = (i_1, ..., i_r)$

$$F^I = F_{i_1} F_{i_2} \cdots F_{i_r}. \tag{1}$$

Then, (Λ^I), (S^I), (Φ^I), and (Ψ^I) are homogeneous bases of **Sym**.

The set of all compositions of a given integer n is equipped with the *reverse refinement order*, denoted $\preccurlyeq$. For example, the compositions J of 5 such that $J \preccurlyeq (2, 1, 2)$ are $(2, 1, 2)$, $(3, 2)$, $(2, 3)$, and (5). The basis (R_I) of *ribbon Schur functions*, originally defined in terms of quasi-determinants in [4], can also be defined by either of the two equivalent equations

$$S^I = \sum_{J \preccurlyeq I} R_J, \qquad R_I = \sum_{J \preccurlyeq I} (-1)^{l(I) - l(J)} S^J, \tag{2}$$

$l(I)$ being the length of the composition I.

The commutative image of a noncommutative symmetric function is given by the algebra morphism $\Lambda_n \mapsto e_n$. Then, $S_n \mapsto h_n$, $\Psi_n \mapsto p_n$, $\Phi_n \mapsto p_n$, and R_I is sent to an ordinary ribbon Schur function, which is denoted by r_I. Ribbon Schur functions have been defined by McMahon (see [12, t. 1, p. 200]) and are denoted in his book by h_I.

Let$\sigma \in \mathbf{S}_n$ be a permutation with descent set $A = \{d_1 < \cdots < d_k\} \subseteq n-1 := \{1, ..., n-1\}$. The *descent composition* $I = C(\sigma)$ is the composition $I = (i_1, ..., i_{k+1})$ of n defined by $i_s = d_s - d_{s-1}$, where $d_0 := 0$ and $d_{k+1} := n$. We also set $I = C(A)$, and conversely, the subset A of $n-1$ associated to a composition I of n is denoted by $A = E(I)$. The sum in the group algebra of all permutations with descent composition I is denoted by D_I. The D_I with $|I| = n$ form a basis of a subalgebra $\mathbf{Z}[\mathbf{S}_n]$, called the *descent algebra* of $\mathbf{S}_n$ [16]. We denote by Σ_n the same algebra, with scalars extended to $\mathbf{C}$. There is an isomorphism of graded vector spaces

$$\alpha\colon \Sigma = \bigoplus_{n \geqslant 0} \Sigma_n \to \mathbf{Sym} = \bigoplus_{n \geqslant 0} \mathbf{Sym}_n \tag{3}$$

such that

$$\alpha(D_I) = R_I \tag{4}$$

for any composition I.

The direct sum Σ can be given an algebra structure, by extending the natural product of its components Σ_n by setting $xy = 0$ for $x \in \Sigma_p$ and $y \in \Sigma_q$ with $p \neq q$. The *internal product*, denoted $*$, on **Sym** is defined by requiring that α be an *anti*-isomorphism. That is, we set

$$F * G = \alpha(\alpha^{-1}(G)\, \alpha^{-1}(F)).$$

One also defines on **Sym** a coproduct Δ by any of the following equivalent conditions:

$$\Delta S_n = \sum_{k=0}^{n} S_k \otimes S_{n-k}, \qquad \Delta \Lambda_n = \sum_{k=0}^{n} \Lambda_k \otimes \Lambda_{n-k} \tag{5}$$

$$\Delta \Phi_n = \Phi_n \otimes 1 + 1 \otimes \Phi_n, \qquad \Delta \Psi_n = \Psi_n \otimes 1 + 1 \otimes \Psi_n. \tag{6}$$

The fundamental property for computing with the internal product is the following splitting formula.

2.1. PROPOSITION [4]. *Let $F_1, F_2, ..., F_r, G \in \mathbf{Sym}$. Then,*

$$(F_1 F_2 \cdots F_r) * G = \mu_r[(F_1 \otimes \cdots \otimes F_r) * \Delta^r G],$$

where in the right-hand side, μ_r denotes the r-fold ordinary multiplication, and $*$ stands for the operation induced on $\mathbf{Sym}^{\otimes n}$ by $*$.

In the commutative case, this formula can be regarded as a particular case of the Mackey tensor product theorem. Some applications and references can be found in [15].

There is a MAPLE package for noncommutative symmetric functions written by Ung [18] which permits us to calculate examples. Some of the calculations in this paper were done with this tool.

3. Cyclic Characters

The irreducible complex characters of a cyclic group $G := \langle g \rangle$ of order n can be described by

$$\chi_n^k : G \to \mathbf{C}^*, \qquad g^i \mapsto \varepsilon^{ik}, \tag{7}$$

where $\varepsilon := \exp(2\pi i/n)$. If G is a transitive cyclic subgroup of a symmetric group, then G is generated by a full cycle and the Frobenius characteristics of the induced characters $\chi_n^k \uparrow \mathbf{S}_n$ are [3]

$$l_n^{(k)} = \frac{1}{n} \sum_{i=1}^{n} \varepsilon^{ik} p_{n \wedge i}^{n/n \wedge i} = \frac{1}{n} \sum_{d \mid n} c(k, d)\, p_d^{n/d}, \tag{8}$$

where $\wedge$ means g.c.d. and $c(—, —)$ denotes the von Sterneck function, i.e., $c(m, k)$ is the sum of mth powers of the primitive kth roots of unity (these sums are sometimes referred to as Ramanujan sums).

A result of Kraskiewicz and Weyman [9] gives the decomposition of $l_n^{(k)}$ into Schur functions or as well into ribbon Schur functions:

$$l_n^{(k)} = \sum_{\mathrm{maj}(I) \equiv k \bmod n} r_I; \tag{9}$$

i.e., the sum is over all ribbons parametrized by compositions I whose major index is congruent to $k \bmod n$ (the major index of a composition I is defined as the sum of the elements of the associated descent set $E(I)$).

We will take this equation as starting point for defining a noncommutative analogue of $l_n^{(k)}$ in the algebra of noncommutative symmetric functions [4]. It will turn out that there are natural noncommutative analogues of the power sums $p_d^{n/d}$ which give an expansion corresponding to (8).

More generally, if G is a cyclic subgroup of a symmetric group $\mathbf{S}_m$ generated by an element g of order n with cycle partition I, then the Frobenius characteristics of the induced characters $\chi_n^k \uparrow \mathbf{S}_m$ are easily described within the formalism of inner plethysm

$$l_I^{(k)} := \frac{1}{n} \sum_{d \mid n} c(k, d)\, \hat{p}_{n/d}^*(p_I). \tag{10}$$

Here $\hat{p}_s^*$ denotes the adjoint Adams operation of inner plethysm which is an algebra homomorphism defined on the generators p_i by

$$\hat{p}_s(p_i) := p_{s \wedge i}^{i/s \wedge i}.$$

The decomposition of these cyclic characters into Schur functions is given in [17]. We reformulate this result in term of ribbon Schur functions.

Recall that the order of g is equal to the l.c.m. of its cycle lengths,

$$n = \text{l.c.m.}(i_1, ..., i_r),$$

if $I = (i_1, ..., i_r)$. Hence we may form

$$I(n) := (n/i_1, 2n/i_1, ..., n/i_2, ..., ni_r/i_r)$$

and define, for any composition J of m,

$$\text{smaj}_{I(n)}(J) := \sum_{u \in E(J)} (I(n))_u \, u. \tag{11}$$

Then the decomposition is

$$l_I^{(k)} = \sum_{\text{smaj}_{I(n)}(J) \equiv k \bmod n} r_J. \tag{12}$$

4. Noncommutative Cyclic Characters

Let $n \in \mathbf{N}$ be given and $\varepsilon := \exp(2\pi i/n)$. For an indeterminate q we define

$$K_n(q) := \sum_{|I| = n} q^{\text{maj}(I)} R_I, \qquad \mathbf{k}(q) := \sum_{n \geqslant 0} (K_n(q)/(n)_q). \tag{13}$$

It is proved in [4] that

$$\mathbf{k}(q) = \overleftarrow{\prod_{k \geqslant 0}} \sigma(q^k) = \cdots \sigma(q^2)\, \sigma(q)\, \sigma(1). \tag{14}$$

Imitating (9) we put

$$L_n^{(k)} := \sum_{\text{maj}(I) \equiv k \bmod n} R_I. \tag{15}$$

The elements $L_n^{(k)}$ will be called *noncommutative cyclic characters.*

Another basis of the subspace spanned by the $L_n^{(k)}$ is given by

$$K_n^{(k)} := K_n(\varepsilon^k) = \sum_i \varepsilon^{ik} L_n^{(i)}, \tag{16}$$

and the transition matrix $\Xi(L, K)$ from the elements $L_n^{(k)}$ to the $K_n^{(j)}$ is the character table of the cyclic group of order n,

$$\Xi(L, K) = (\varepsilon^{ik})_{i,k}.$$

The inverse transformation is described by

$$\Xi(K, L) = \Xi(L, K)^{-1} = \frac{1}{n} (\varepsilon^{-ik})_{k,i},$$

i.e.

$$L_n^{(k)} = \frac{1}{n} \sum_i \varepsilon^{-ik} K_n^{(i)}, \tag{17}$$

corresponding to (8).

If ε^k is a primitive nth root of unity, i.e., $n \wedge k = 1$, then $(1/n)\, K_n^{(k)}$ is an idempotent corresponding to Klyachko's idempotent in the descent algebra [8]. In particular, $K_n^{(k)}$ is primitive for Δ [4].

If ε^k is not a primitive root of unity, then $K_n^{(k)}$ is a product of primitive elements, for we have the following.

4.1. Proposition. *Let ζ be any rth root of unity. Let $n = ar + b$ with $a, b \in \mathbf{N}$, $b < r$. Then*

$$K_n(\zeta) = K_r(\zeta)^a \, K_b(\zeta).$$

Proof. As ζ is not assumed to be a primitive rth root of unity, it is sufficient to prove

$$K_{r+s}(\zeta) = K_r(\zeta)\, K_s(\zeta).$$

Let now I be a composition of n and $A := E(I) \subseteq \underline{n-1}$ the descent set associated to I. If one puts

$$B := \{a \in A \mid a < r\}, \qquad C := \{a - r \in A \mid a \geqslant r\} \backslash \{0\},$$

then B gives a composition J of r and C a composition J' of $n - r$ such that

$$R_J R_{J'} = R_I + R_{I'},$$

for some composition I' of n, and $\mathrm{maj}(J) + \mathrm{maj}(J') \equiv \mathrm{maj}(I) \bmod r$. Moreover, $K_r(1)\, K_s(1) = S_1^r S_1^s = S_1^{r+s} = K_{r+1}(1)$ shows that each R_I in $K_r(\zeta)\, K_s(\zeta)$ comes from exactly one pair R_J, $R_{J'}$ in the respective factors. ∎

In the sequel we need a description of the action of $K_n(q)$ on a product of primitive elements. In order to describe this we need a generalization of the major index [8, 1]. If $I := (i_1, ..., i_r)$ is a composition of n and $\rho \in \mathbf{S}_r$, then

$$\mathrm{maj}_I(\rho) := \sum_{j \in \mathrm{Des}(\rho)} (i_{\rho 1} + i_{\rho 2} + \cdots + i_{\rho j}).$$

4.2. Proposition. *Let $I := (i_1, ..., i_r)$ be a composition of n and F_{i_j} ($j = 1, ..., r$) be a primitive, homogeneous element of degree i_j of* **Sym**. *Then*

$$K_n(q) * F^I = (q)_n \sum_{\rho \in \mathbf{S}_r} \frac{q^{\mathrm{maj}_I(\rho)}}{(1-q^{i_{\rho 1}})(1-q^{i_{\rho 1}+i_{\rho 2}}) \cdots (1-q^{i_{\rho 1}+ \cdots + i_{\rho r}})} F^{\rho I}.$$

Proof. We will compute $\mathbf{k}(q) * F^I$, taking into account the factorization (14) of $\mathbf{k}(q)$. Let $\mathbf{k}_N(q) := \sigma(q^N)\, \sigma(q^{N-1}) \cdots \sigma(q)\, \sigma(1)$. Then by Proposition 2.1

$$\mathbf{k}_N(q) * F^I = \sum_{f : \underline{r} \to \underline{N}} q^{N_f} F^{I_f},$$

where

$$N_f := N \cdot |f^{-1}(1)| + (N-1) \cdot |f^{-2}(2)| + \cdots + 1 \cdot |f^{-1}(N-1)|,$$

and $I_f := (i_{\rho 1}, ..., i_{\rho r})$ is a permutation ρI of I which arises as

$$I_f = (\underbrace{i_{\rho 1}, ..., i_{\rho s_1}}_{\in f^{-1}(1)}, \underbrace{i_{\rho(s_1+1)}, ..., i_{\rho(s_1+s_2)}}_{\in f^{-1}(2)}, ...)$$

and

$$\rho 1 < \rho 2 < \cdots < \rho s_1,\ \rho(s_1+1) < \rho(s_1+2) < \cdots < \rho(s_1+s_2) < \cdots.$$

Hence it remains to see which functions f give rise to a fixed permutation $\rho \in \mathbf{S}_r$ and which factor they contribute. As I_f is clearly nondecreasing on the blocks $f^{-1}(u)$, these are precisely those f that are nondecreasing on the blocks given by the descent set of ρ.

On the other hand, I_f can be written as a linear combination with nonnegative integer coefficients of the elements:

$$g_j := i_{\rho 1} + \cdots + i_{\rho j}, \qquad 1 \leqslant j \leqslant r-1.$$

But the coefficients of g_j, $j \in \mathrm{Des}(\rho)$, must be positive. Hence the sum of all terms for which f corresponds to a given ρ is

$$\sum_{\substack{a_j \in \mathbf{N} \\ \alpha_j > 0 \text{ for } j \in \mathrm{Des}(\rho)}} q^{a_1 g_1 + a_2 g_2 + \cdots} = q^{\mathrm{maj}_I(\rho)} \sum_{a_j \in \mathbf{N}} q^{a_1 g_1 + a_2 g_2 + \cdots}$$
$$= \frac{q^{\mathrm{maj}_I(\rho)}}{(1-q^{g_1}) \cdots (1-q^{g_r})}. \quad \blacksquare$$

4.3. Corollary. *If $n = ra$, then*

$$K_n(q) * (F_r)^a = \frac{(q)_n}{(1-q^r)^a} (F_r)^a.$$

Proof. As $I = (r, r, ..., r)$, we have $\rho I = I$ and everything follows from Proposition 4.2 and

$$\sum_{\rho \in \mathbf{S}_a} \frac{t^{\mathrm{maj}(\rho)}}{(1-t)(1-t^2) \cdots (1-t^a)} = \frac{1}{(1-t)^a}$$

a well-known identity of McMahon [12]. $\blacksquare$

Taking the commutative image of Proposition 4.2 and the specialization $p_k \to 1$ $(k \geqslant 1)$, we see that this last identity can be generalized to the following.

4.4. Proposition. *For each composition $I := (i_1, ..., i_r)$ of n one has*

$$\sum_{\rho \in \mathbf{S}_r} \frac{q^{\mathrm{maj}_I(\rho)}}{(1-q^{i_{\rho 1}})(1-q^{i_{\rho 1} + i_{\rho 2}}) \cdots (1-q^{i_{\rho 1} + \cdots + i_{\rho r}})}$$
$$= \frac{1}{(1-q^{i_1})(1-q^{i_2}) \cdots (1-q^{i_r})}.$$

Specializing q to certain roots of unity gives the following.

4.5. Corollary. *Let ε be a primitive mth root of unity, $n = ra$ and let F_r be primitive homogeneous of degree r. Then*:

(i) $K_n(\varepsilon) * (F_r)^a = 0$, *if $m = n$, $r \neq n$.*

(ii) $K_n(\varepsilon) * (F_r)^a = a!\, r^a (F_r)^a$, *if $m = r$.*

(iii) *If ε, ζ are two primitive nth roots of unity, then $K_n(\varepsilon) * K_n(\zeta) = nK_n(\zeta)$.*

5. The Algebra Generated by the $L_n^{(k)}$

Consider the vector space $\mathscr{C}_n$ with basis $(K_n^{(j)})$ or $(L_n^{(k)})$. We will show that $\mathscr{C}_n$ is a subalgebra w.r.t. the internal product $*$ and compute the structure constants for these two bases.

5.1. Proposition.

$$K_n^{(k)} * K_n^{(l)} = \begin{cases} (n/d)!\, d^{n/d} K_n^{(l)} & \text{if} \quad n \wedge k = n \wedge l = d \\ 0 & \text{otherwise.} \end{cases}$$

In particular,

$$\mathscr{C}_n := \langle\!\langle K_n^{(k)} \mid k = 1, \ldots n \rangle\!\rangle = \langle\!\langle L_n^{(k)} \mid k = 1, \ldots n \rangle\!\rangle$$

is a subalgebra of the algebra of noncommutative symmetric functions with respect to the internal product $*$ which is noncommutative for $n \geqslant 3$.

Proof. If we set $G := K_{n \wedge k}(\varepsilon^k)$, $F := K_{n \wedge l}(\varepsilon^l)$, then according to Proposition 2.1

$$\begin{aligned} K_n^{(k)} * K_n^{(l)} = G^{n/n \wedge k} * F^{n/n \wedge l} &= \mu_{n/n \wedge k}((G \otimes \cdots \otimes G) * \Delta^{n/n \wedge k}(F^{n/n \wedge l}) \\ &= \mu_{n/n \wedge k}((G \otimes \cdots \otimes G)\, \Delta^{n/n \wedge k}(F)^{n/n \wedge l}) \end{aligned}$$

As F is primitive, it is easy to expand the coproduct. Hence we have to consider a sum of products of the form $G * F^s$. For degree reasons, this expression is zero if $n \wedge k = \deg(G) < \deg(F) = n \wedge l$. By 4.3 it also vanishes if $s > 1$; i.e., $n \wedge k > n \wedge l$. Hence, the product itself is zero if $n \wedge k \neq n \wedge l$. But if $n \wedge k = n \wedge l$, the product is a sum of identical terms

$$(G * F)^{n/n \wedge l} = (n \wedge l)^{n/n \wedge l}\, F^{n/n \wedge l},$$

and the multiplicity of this term is $(n/n \wedge l)!$. The proposition follows. ∎

From this result the computation of $L_n^{(k)} * L_n^{(l)}$ is straightforward,

$$L_n^{(k)} * L_n^{(l)} = \frac{1}{n^2} \sum_{i,j} \varepsilon^{-ik-jl} K_n^{(i)} * K_n^{(j)} = \frac{1}{n^2} \sum_{i,j:\, n \wedge i = n \wedge j} \varepsilon^{-ik-jl} f_{j,n} K_n^{(j)},$$

where $f_{j,n} := (n/n \wedge j)!\, (n \wedge j)^{n/n \wedge j}$. From this we get

$$\begin{aligned} &\frac{1}{n^2} \sum_{i,j:\, n \wedge i = n \wedge j} \varepsilon^{-ik-jl} f_{j,n} \sum_m \varepsilon^{jm} L_n^{(m)} \\ &\quad = \sum_m \left(\frac{1}{n^2} \sum_{i,j:\, n \wedge i = n \wedge j} \varepsilon^{-ik} \varepsilon^{j(m-l)} f_{j,n} \right) L_n^{(m)}. \end{aligned}$$

But an easy computation shows that the expression in braces is just a scalar product of symmetric functions, namely,

$$\langle l_n^{(k)}, l_n^{(m-l)} \rangle = \langle l_n^{(-k)}, l_n^{(m-l)} \rangle,$$

whence we have the following.

5.2. Theorem.

$$L_n^{(k)} * L_n^{(l)} = \sum_{m=1}^{n} \langle l_n^{(k)}, l_n^{(m-l)} \rangle \, L_n^{(m)}.$$

The commutative image gives the resolution of a Kronecker product of cyclic representations into a direct sum of cyclic representations. Taking into account the results of [9], this can also be interpreted as a combinatorial formula for the decomposition into irreducibles.

6. A Combinatorial Consequence

Theorem 5.2 has a nice interpretation in terms of permutations, if we apply the anti-isomorphism α given in (3). As

$$D_n^{(k)} := \alpha^{-1}(L_n^{(k)}) = \sum_{\substack{|I|=n \\ \mathrm{maj}(I) \equiv k \bmod n}} D_I,$$

Proposition 5.2 immediately translates to

$$D_n^{(l)} D_n^{(k)} = \sum_{m=1}^{n} \langle l_n^{(k)}, l_n^{(m-l)} \rangle \, D_n^{(m)}.$$

Expanding the product and comparing the coefficients we arrive at the following.

6.1. Proposition. *Let $\pi \in \mathbf{S}_n$ with $\mathrm{maj}(\pi) \equiv m \bmod n$. Then the number of elements $\sigma, \tau \in \mathbf{S}_n$ with $\pi = \sigma\tau$ and $\mathrm{maj}(\sigma) \equiv l$, $\mathrm{maj}(\tau) \equiv k \bmod n$ only depends on m, k, l and is equal to the scalar product*

$$\langle l_n^{(k)}, l_k^{(m-l)} \rangle.$$

Let us give an example in $\mathbf{S}_4$. We have

$$l_4^{(0)} = (p_1^4 + p_2^2 + 2p_4)/4, \qquad l_4^{(1)} = l_4^{(3)} = (p_1^4 - p_2^2)/4,$$
$$l_4^{(2)} = (p_1^4 + p_2^2 - 2p_4)/4.$$

The scalar products are

	$l_4^{(0)}$	$l_4^{(1)}$	$l_4^{(2)}$
$l_4^{(0)}$	3	1	1
$l_4^{(1)}$	1	2	1
$l_4^{(2)}$	1	1	3

Consider $m=2$, $k=2$, $l=1$; $\pi := 2314 \in S_4$ has major index congruent to $2 \bmod 4$ and there is only one factorization $\pi = \sigma\tau = (2134)\,(1324)$ such that $\mathrm{maj}(\sigma) \equiv 1 \bmod 4$ and $\mathrm{maj}(\tau) \equiv 2 \bmod 4$.

7. Rectangles

The previous sections dealt with the case of a transitive cyclic subgroup of a symmetric group. This section contains some comments on the general case. But it turns out that the results are not as nice as before. Imitating (12) we define

$$K_I(q) := \sum_{|J|=n} q^{\mathrm{smaj}_{I(n)}(J)} R_J, \tag{18}$$

and generalized cyclic characters

$$L_I^{(k)} := \sum_{\mathrm{smaj}_{I(n)}(J) \equiv k \bmod n} R_J. \tag{19}$$

It is easy to see that there is a factorization

$$K_I(q) = K_{i_1}(q^{n/i_1})\, K_{i_2}(q^{n/i_2}) \cdots \qquad \bmod q^n - 1 \tag{20}$$

and as in Section 4 one has the relations

$$K_I^{(k)} := K_I(\varepsilon^k) = \sum_i \varepsilon^{ik} L_I^{(i)}, \qquad L_I^{(k)} = \frac{1}{n} \sum_i \varepsilon^{-ik} K_I^{(i)}. \tag{21}$$

But it turns out that the subspaces generated by the $(L_I^{(k)})$ or $(K_I^{(j)})$ are no longer subalgebras in general.

7.1. Example. Consider $\mathbf{S}_5$ and $I := (3, 2)$, i.e., $n := 6$. Let $\varepsilon := \exp(2\pi i/6)$. Then

$$K_I^{(1)} := K_3(\varepsilon^2)\, K_2(-1)$$
$$K_I^{(5)} := K_3(\varepsilon^4)\, K_2(-1)$$

$$K_I^{(2)} := K_3(\varepsilon^4)\, K_1(1)\, K_1(1)$$

$$K_I^{(4)} := K_3(\varepsilon^2)\, K_1(1)\, K_1(1)$$

$$K_I^{(3)} := K_1(1)\, K_1(1)\, K_1(1)\, K_2(-1)$$

$$K_I^{(6)} := K_1(1)^5.$$

The multiplication table $K_I^{(j)} * K_I^{(k)}$ is contained in

$j \backslash k$	1	5	2	4	3	6	a
1	$6K_I^{(1)}$	$6K_I^{(5)}$	0	0	0	0	0
5	$6K_I^{(1)}$	$6K_I^{(5)}$	0	0	0	0	0
2	0	0	$6K_I^{(2)}$	$6K_I^{(4)}$	$6(1-\varepsilon^2)\,a$	0	$6a$
4	0	0	$6K_I^{(2)}$	$6K_I^{(4)}$	$6(1-\varepsilon^4)\,a$	0	$6a$
3	0	0	0	0	$12K_I^{(3)}$	0	0
6	0	0	0	0	0	$720K_I^{(6)}$	0
a	0	0	0	0	$12a$	0	0

where $a := K_1(1)\, K_2(-1)\, K_1(1)^2 - K_2(-1)\, K_1(1)^3$. This table can be easily computed with (4.2) and (4.5). We will make one particular case explicit which shows why the space generated by the $K_I^{(j)}$ is not a subalgebra. Let $F_1 := K_1(1)$, $F_2 := K_2(-1)$, $F_3 := K_3(\varepsilon^2)$. Then

$$\begin{aligned} K_I^{(2)} * K_I^{(3)} &= 6(F_3 F_1^2) * (F_1^3 F_2) \\ &= 6(F_3 * (F_1 F_2))(F_1 * F_1)^2 \\ &= 6(F_3 * (F_1 F_2))\, F_1^2. \end{aligned}$$

but by (4.2) $F_3 * (F_1 F_2)$ is equal to

$$\begin{aligned} &(q)_3 \left(\frac{1}{(1-q)(1-q^3)}\, F_1 F_2 + \frac{q^2}{(1-q^2)(1-q^3)}\, F_2 F_1 \right) \\ &= (1-q^2)\, F_1 F_2 + (q^2 - q^3)\, F_2 F_1. \end{aligned}$$

Hence,

$$K_I^{(2)} * K_I^{(3)} = 6(1-\varepsilon^2)(F_1 F_2 - F_2 F_1)\, F_1^2 = 6(1-\varepsilon^2)\, a.$$

In the preceding example we arranged the indices in a particular way:

For each divisor d of n we collected the $1 \leqslant i \leqslant n$ with $i \wedge n = d$ in blocks and arranged the numbers within the blocks and the blocks itself in increasing order. As for products of primitive elements we have

$$F^J * G^{J'} = 0 \qquad \text{if the length of } J \text{ is greater thant that of } J';$$

this shows that the table is upper block triangular. In fact, the block diagonal elements can be easily computed,

$$K_I^{(j)} * K_I^{(k)} = c_I K_I^{(k)}, \tag{22}$$

where c_I is the order of the centralizer of an element with cycle partition I. In particular, the $K_I^{(k)}$ are quasi-idempotent.

But, as the example shows, there might be nonzero elements above the diagonal (which belong to the radical of **Sym**). In (7.1) these elements were nilpotent of order 2 and the subalgebra generated by the $K_I^{(j)}$ has a basis containing the $K_I^{(j)}$, together with an element a, $a * a = 0$.

There is one particular case where the multiplication table is in fact block diagonal: the case $I := (r^s)$, $m = rs$, a rectangular partition.

7.2. Example. Let $I := (4, 4)$ and $\varepsilon := \exp(2\pi i/4) = i$, $\zeta := \exp(2\pi i/8)$. Then

$$K_I^{(1)} = K_4(i)^2 = K_8(\zeta^2)$$
$$K_I^{(3)} = K_4(-i)^2 = K_8(\zeta^6)$$
$$K_I^{(2)} = K_2(-1)^4 = K_8(\zeta^4)$$
$$K_I^{(4)} = K_1(1)^8 = K_8(\zeta^8)$$

$j\backslash k$	1	3	2	4
1	$32K_I^{(1)}$	$32K_I^{(3)}$	0	0
3	$32K_I^{(1)}$	$32K_I^{(3)}$	0	0
2	0	0	$384K_I^{(2)}$	0
4	0	0	0	$(8!)\,K_I^{(4)}$

This example clearly shows that for $I := (r^s)$ the space $\mathscr{C}_I$ generated by $K_I^{(j)}$ is a subalgebra of $\mathscr{C}_n$. We just record the multiplication formulas.

7.3. Proposition. *Let $m := rs$, $I := (r^s)$. Then*

(i) $K_I^{(k)} * K_I^{(l)} = \begin{cases} (n/d)!\, d^{n/d} K_I^{(l)} & \text{if } \; r \wedge k = r \wedge l = d, \\ 0 & \text{otherwise.} \end{cases}$

(ii) $L_I^{(k)} * L_I^{(l)} = \Sigma_{i=1}^r \langle l_I^{(k)}, l_I^{(m-l)} \rangle\, L_I^{(m)}$.

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References

1. D. Blessenohl and H. Laue, On Witt's dimension formula and a theorem of Klyachko, *Bull. Austral. Math. Soc.* **40** (1989), 49–57.
2. G. Duchamp, D. Krob, B. Leclerc, and J.-Y. Thibon, Déformations de projecteurs de Lie, *C.R. Acad. Sci. Paris* **319** (1994), 909–914.
3. H. O. Foulkes, Characters of symmetric groups induced by characters of cyclic subgroups, in Combinatorics *in* "Proceedings Conf. Combin. Math. Inst. Oxford, 1972, Inst. Math. Appl., Southend-on-Sea, 1972," pp. 141–154.
4. I. M. Gelfand, D. Krob, A. Lascoux, B. Leclerc, V. S. Retakh, and J.-Y. Thibon, Noncommutative symmetric functions, *Adv. in Math.* **112** (1995), 218–348.
5. I. M. Gelfand and V. S. Retakh, Determinants of matrices over noncommutative rings, *Funct. Anal. Appl.* **25** (1991), 91–102.
6. I. M. Gelfand and V. S. Retakh, A theory of noncommutative determinants and characteristic functions of graphs, *Funct. Anal. Appl.* **26** (1992), 1–20; Publ. Lacim, UQAM, Montreal, Vol. 14, pp. 1–26.
7. I. Gessel, Multipartite *P*-partitions and inner product of skew Schur functions, *Contemp. Math.* **34** (1984), 289–301.
8. A. A. Klyachko, Lie elements in the tensor algebra, *Siberian Math. J.* **15** (1974), 1296–1304.
9. W. Kraskiewicz and J. Weyman, Algebra of invariants and the action of a Coxeter element, preprint, Math. Inst. Univ. Copernic, Toruń, Poland.
10. A. Lascoux, B. Leclerc, and J.-Y. Thibon, Green polynomials and Hall–Littlewood functions at roots of unity, *European J. Combin.* **15** (1994), 173–180.
11. I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Oxford Math. Monographs, Oxford Univ. Press, Oxford, 1979.
12. P. A. MacMahon, "Combinatory Analysis," Cambridge Univ. Press, Cambridge 1915, 1916; Chelsea, New York, 1960.
13. B. Mielnik and J. Plebański, Combinatorial approach to Baker–Campbell–Hausdorff exponents, *Ann. Inst. H. Poincaré Sect. A* **12** (1970), 215–254.
14. C. Reutenauer, "Free Lie Algebras," Oxford Univ. Press, Oxford, 1993.
15. T. Scharf and J.-Y. Thibon, A Hopf algebra approach to inner plethysm, *Adv. in Math.* **104** (1994), 30–58.
16. L. Solomon, A Mackey formula in the group ring of a Coxeter group, *J. Algebra* **41** (1976), 255–268.
17. J. R. Stembridge, On the eigenvalues of representations of reflection groups and wreath products, *Pacific J. Math* **140** (1989), 353–396.
18. B. C. V. Ung, "NCSF, a MAPLE Package for Noncommutative Symmetric Functions," Preprint, Université de Marne-la-Vallée, 1994.